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On the Existence of Positive Solutions for Hemivariational Inequalities Driven by the p-Laplacian*

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Abstract. We study nonlinear elliptic problems driven by the p-Laplacian and with a nonsmooth locally Lipschitz potential (hemivariational inequality). We do not assume that the nonsmooth potential satisfies the Ambrosetti–Rabinowitz condition. Using a variational approach based on the nonsmooth critical point theory, we establish the existence of at least one smooth positive solution.

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1. Introduction

The study of the existence of positive solutions for elliptic equations has focused primarily on semilinear (i.e. $p=2$) problems with a smooth potential function (i.e. a continuous right hand side nonlinearity). The techniques used are based on topological degree theory, variational methods (i.e. critical point theory) and on the method of upper and lower solutions coupled with monotone iterative techniques. We refer to the works of Alves and Miyagaki [1], Amann [2], Ambrosetti and Rabinowitz [3], Brezis and Turner [4], Brown and Budin [5], de Figueiredo [11], Schechter [27] and Zhou [29]. One condition often used in the context of variational methods is the so-called Ambrosetti–Rabinowitz condition (see Alves and Miyagaki [1] and Ambrosetti and Rabinowitz [3]). This condition says the following:

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*Revised version.

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Let $f(t,\zeta)$ be the right hand side nonlinearity and let

$$F(z,\zeta)=\int_0^{\zeta} f(z,r)\mathrm{d}r$$

be the corresponding potential function. There exist $\mu>2$ (recall that in the semilinear case $p=2$) and $M>0$, such that

$$0<\mu F(z,\zeta)\leqslant \zeta f(z,\zeta) \quad \text{for a.a. } z\in Z \text{ and all } |\zeta|>M.$$

Integrating this condition, we obtain $c_1, c_2>0$, such that

$$c_1|\zeta|^{\mu}-c_2\leqslant F(z,\zeta) \quad \text{for a.a. } z\in Z \text{ and all } \zeta\in\mathbb{R}.$$

From this growth condition on F, we see that

$$\lim_{|\zeta|\to+\infty}\frac{F(z,\zeta)}{\zeta^2}=+\infty,$$

i.e. the potential function is superquadratic or equivalently f is superlinear. In this setting, we can realize the Mountain Pass Geometry and eventually apply the Mountain Pass Theorem. However, in several physical applications the nonlinearity $f(z,\cdot)$ is asymptotically linear, a requirement which is incompatible with the Ambrosetti–Rabinowitz condition. In this paper we study problems driven by the p-Laplacian and with a nonsmooth, locally Lipschitz potential (hemivariational inequalities) which do not satisfy the Ambrosetti–Rabinowitz condition or any of its recent generalizations (see Costa and Magalhaes [8] and Schechter [27]). We look for positive solutions of such problems. The study of positive solutions for nonlinear nonsmooth elliptic problems is lagging behind and only recently there has been the work of Gasiński and Papageorgiou [15], based on different assumptions and techniques.

Let $Z\subseteq\mathbb{R}^N$ be a bounded domain with a $C^{1,\alpha}$-boundary Γ (where $0<\alpha<1$). The problem under consideration is the following:

$$\begin{cases} -\mathrm{div}\big(\|\nabla x(z)\|_{\mathbb{R}^N}^{p-2}\nabla x(z)\big)\in\partial j(z,x(z)) & \text{for a.a. } z\in Z\\ x|_{\Gamma}=0. \end{cases} \tag{1.1}$$

Here $p\in(1,+\infty)$, j is a locally Lipschitz integrand and $\partial j(z,\zeta)$ denotes the subdifferential of $j(z,\cdot)$ in the sense of Clarke [7] (generalized subdifferential; see Section 2). Problems like (1.1) are known in the literature as 'hemivariational inequalities' and arise in mechanics when one wants to consider more realistic models with nonsmooth and nonconvex energy functionals. For several such applications we refer to the book of Naniewicz and Panagiotopoulos [25]. Also problem (1.1) includes as a special case equations with discontinuous nonlinearities (see Chang [6] and Kourogenis and Papageorgiou [19]). The mathematical theory of hemivariational inequalities can be traced in the recent works of Gasiński and Papageorgiou [12–14], Goeleven et al. [16], Motreanu and Panagiotopoulos [23], Motreanu and Varga [24] and Radulescu [26].

2. Mathematical Background

Our approach is variational and it is based on the nonsmooth critical point theory (see Chang [6] and Kourogenis and Papageorgiou [19]), which in turn uses the subdifferential theory for locally Lipschitz functions due to Clarke [7]. For easy reference, in this section we recall some basic definitions and facts from these theories, which we shall need in the sequel. We also recall some basic results about the spectrum of $(-\Delta_p, W_0^{1,p}(Z))$ (i.e. of the negative p-Laplacian with Dirichlet boundary condition).

Let X be a Banach space and X^* its topological dual. By $\|\cdot\|_X$ we will denote the norm of X and by $\langle\cdot,\cdot\rangle_X$ the duality brackets for the pair (X, X^*). A function $\varphi\colon X \longmapsto \mathbb{R}$ is said to be *locally Lipschitz*, if for every $x \in X$, there exists a neighbourhood U of x and a constant $k_U > 0$ depending on U, such that $|\varphi(z) - \varphi(y)| \leqslant k_U \|z - y\|_X$ for all $z, y \in U$. From convex analysis we know that a proper, convex and lower semicontinuous function $\psi\colon X \longmapsto \overline{\mathbb{R}} \stackrel{df}{=} \mathbb{R} \cup \{+\infty\}$ is locally Lipschitz in the interior of its effective domain $\operatorname{dom} \psi \stackrel{df}{=} \{x \in X\colon \psi(x) < +\infty\}$ (see Denkowski et al. [9, Proposition 5.2.10, p. 532]). In analogy with the directional derivative of a convex function, we define the *generalized directional derivative* of a locally Lipschitz function $\varphi\colon X \longrightarrow \mathbb{R}$ at $x \in X$ in the direction $h \in X$, by

$$\varphi^0(x;h) \stackrel{df}{=} \limsup_{\substack{x' \to x \\ t \searrow 0}} \frac{\varphi(x' + th) - \varphi(x')}{t}.$$

The function $X \ni h \longmapsto \varphi^0(x;h) \in \mathbb{R}$ is sublinear, continuous and by the Hahn–Banach theorem it is the support function of a nonempty, convex and w^*-compact subset of X^*, defined by

$$\partial\varphi(x) \stackrel{df}{=} \left\{x^* \in X^*\colon \langle x^*, h\rangle_X \leqslant \varphi^0(x;h) \text{ for all } h \in X\right\}.$$

The multifunction $X \ni x \longmapsto \partial\varphi(x) \in 2^{X^*} \setminus \{\emptyset\}$ is called *the Clarke* (or *generalized*) *subdifferential* of φ at x. If $\varphi, \psi\colon X \longmapsto \mathbb{R}$ are locally Lipschitz functions, then $\partial(\varphi + \psi)(x) \subseteq \partial\varphi(x) + \partial\psi(x)$ and $\partial(t\varphi)(x) = t\partial\varphi(x)$ for all $t \in \mathbb{R}$ and all $x \in X$.

If $\varphi\colon X \longmapsto \mathbb{R}$ is continuous, convex (thus locally Lipschitz as well), then for all $x \in X$, the generalized subdifferential introduced above coincides with the subdifferential of φ in the sense of convex analysis, given by

$$\partial\varphi(x) \stackrel{df}{=} \left\{x^* \in X^*: \langle x^*, h\rangle_X \leqslant \varphi'(x;h) \text{ for all } h \in X\right\},$$

where $\varphi'(x;h)$ is the usual directional derivative of φ at x in the direction h. If φ is strictly differentiable at x (in particular if φ is continuously Gateaux differentiable at x), then $\partial\varphi(x) = \{\varphi'(x)\}$.

A point $x \in X$ is a critical point of the locally Lipschitz function φ, if $0 \in \partial\varphi(x)$. If $x \in X$ is a critical point, the value $c = \varphi(x)$ is a critical value of φ. It is easy to check that, if $x \in X$ is a local extremum of φ (i.e. a local minimum or a local maximum), then $0 \in \partial\varphi(x)$ (i.e. $x \in X$ is a critical point).

In the classical (smooth) theory, a compactness-type condition, known as the Palais–Smale condition and its extension, the Cerami condition play a crucial role. In the present nonsmooth setting the Cerami condition (which we shall use in what follows), takes the following form:

A locally Lipschitz function φ: $X\to\mathbb{R}$ satisfies the *nonsmooth Cerami condition at level* $c\in\mathbb{R}$ if any sequence $\{x_n\}_{n\geqslant 1}\subseteq X$, such that $\varphi(x_n)\longrightarrow c$ is and

$$\big(1+\|x_n\|_X\big)m_\varphi(x_n)\longrightarrow 0 \quad \text{as } n\to+\infty,$$

where

$$m_\varphi(x_n)\stackrel{df}{=}\min\big\{\|x^*\|_{X^*}:\ x^*\in\partial\varphi(x_n)\big\}$$

has a strongly convergent subsequence. If this condition is satisfied at every level $c\in\mathbb{R}$, then we simply say that φ satisfies the *nonsmooth Cerami condition*.

Our analysis of problem (1.1) involves λ_1, the principle eigenvalue of the negative p-Laplacian with Dirichlet boundary condition $(-\Delta_p, W_0^{1,p}(Z))$. So we need to recall a few known facts about the spectrum of $(-\Delta_p, W_0^{1,p}(Z))$. For this purpose consider the following nonlinear eigenvalue problem:

$$\begin{cases} -\operatorname{div}\big(\|\nabla x(z)\|_{\mathbb{R}^N}^{p-2}\nabla x(z)\big)=\lambda|x(z)|^{p-2}x(z) \quad \text{for a.a. } z\in Z\\ x|_\Gamma=0, \end{cases} \tag{2.1}$$

with $p\in(1,+\infty)$. The least real number λ for which problem (2.1) has a nontrivial solution is called the first eigenvalue of $(-\Delta_p, W_0^{1,p}(Z))$ and is denoted by λ_1. It is known that $\lambda_1>0$, it is isolated and simple (i.e. the associated eigenspace is one-dimensional). Moreover, we have a variational characterization of λ_1 via the Rayleigh quotient, namely

$$\lambda_1=\min_{\substack{x\in W_0^{1,p}(Z)\\ x\neq 0}}\frac{\|\nabla x\|_p^p}{\|x\|_p^p} \tag{2.2}$$

and the minimum is attained at the normalized eigenfunction u_1. Remark that, if u_1 minimizes the Rayleigh quotient, then so does $|u_1|$ and so it follows that u_1 does not change sign in Z. Thus we may assume that $u_1(z)\geqslant 0$ on Z. In fact, we have that $u_1(z)>0$ for almost all $z\in Z$ and from the nonlinear regularity theory, we have that $u_1\in C^{1,\beta}(\overline{Z})$ with some $\beta\in(0,1)$. For details we refer to Lindqvist [22] and the references therein. The Lusternik–Schnirelmann theory gives us, in addition to $\lambda_1>0$, a whole strictly increasing sequence of eigenvalues $0<\lambda_1<\lambda_2<\cdots,\ \lambda_n\longrightarrow+\infty$, known as the *Lusternik–Schnirelmann* or *variational eigenvalues* of $(-\Delta_p, W_0^{1,p}(Z))$. If $p=2$, these are all the eigenvalues. If $p\neq 2$, we do not know if this is the case.

The following result is a nonsmooth version of the Mountain Pass Theorem and can be found in Kourogenis and Papageorgiou [19].

THEOREM 2.1. *If* $\varphi\colon X \longrightarrow \mathbb{R}$ *is locally Lipschitz functional,* $x_0, x_1 \in X$, *there exists a bounded open neighbourhood* U *of* x_0, *such that* $x_1 \in X \setminus U$, $\max\{\varphi(x_0), \varphi(x_1)\} < \inf_{\partial U} \varphi$ *and* φ *satisfies the nonsmooth Cerami condition at level c, where*

$$c \stackrel{df}{=} \inf_{\gamma \in \Gamma_0} \max_{t \in [0,1]} \varphi\big(\gamma(t)\big),$$

where

$$\Gamma_0 \stackrel{df}{=} \big\{\gamma \in C\big([0,1]; X\big) \colon \ \gamma(0) = x_0, \ \gamma(1) = x_1\big\},$$

then c *is a critical value of* φ *and* $c \geqslant \inf_{\partial U} \varphi$.

For a function $x \in W_0^{1,p}(Z)$, we define

$$x^- \stackrel{df}{=} \max\{-x, 0\}$$

$$x^+ \stackrel{df}{=} \max\{x, 0\}.$$

We know that $x^-, x^+ \in W_0^{1,p}(Z)$ (see Denkowski et al. [9, Proposition 3.9.29, p. 348]).

3. Positive Solutions

Our hypotheses on the nonsmooth potential function j are the following:
$\underline{H(j)}$ $j\colon Z \times \mathbb{R} \longrightarrow \mathbb{R}$ is a function, such that

(i) for every $\zeta \in \mathbb{R}$, $j(\cdot, \zeta)$ is measurable;
(ii) for almost all $z \in Z$, $j(z, \cdot)$ is locally Lipschitz with $L^\infty(Z)$-constant and $j(z, 0) = 0$;
(iii) for almost all $z \in Z$ and all $u \in \partial j(z, 0)$, we have $u \geqslant 0$;
(iv) for almost all $z \in Z$ and all $u(\zeta) \in \partial j(z, \zeta)$, the function $\zeta \longmapsto \dfrac{u(\zeta)}{\zeta^{p-1}}$ is nondecreasing on $(0, +\infty)$;
(v) there exist functions $\vartheta_0, \vartheta_1 \in L^\infty(Z)$, such that

$$\vartheta_0(z) \leqslant \lambda_1 \leqslant p\vartheta_1(z) \leqslant p\lambda_1 \quad \text{for a.a. } z \in Z,$$

with strict inequalities on sets of positive measures and

$$\limsup_{\zeta \to 0^+} \frac{pj(z, \zeta)}{\zeta^p} \leqslant \vartheta_0(z) \quad \text{and} \quad \lim_{\zeta \to +\infty} \frac{u(z, \zeta)}{\zeta^{p-1}} = \vartheta_1(z)$$

uniformly for almost all $z \in Z$, where $u(z, \zeta) \in \partial j(z, \zeta)$.

Remark 3.1. Note that the assumption

$$\vartheta_1(z) \leqslant \lambda_1 \quad \text{for a.a. } z \in Z$$

(see hypothesis $H(j)(v)$) implies in particular that

$$\lambda_1(\vartheta_1) > 1,$$

where $\lambda_1(\vartheta_1)$ is the first eigenvalue of the weighted eigenvalue problem

$$\begin{cases} -\operatorname{div}\big(\|\nabla x(z)\|_{\mathbb{R}^N}^{p-2}\nabla x(z)\big) = \lambda\vartheta_1(z)|x(z)|^{p-2}x(z) & \text{for a.a. } z\in Z \\ x|_{\Gamma} = 0 \end{cases}$$

(see Denkowski et al. [10, Definition 3.1.49, p. 342]).

We introduce the truncation function $\tau\colon \mathbb{R} \longrightarrow \mathbb{R}_+$, defined by

$$\tau(\zeta) \stackrel{df}{=} \begin{cases} \zeta & \text{if } \zeta > 0 \\ 0 & \text{if } \zeta \leqslant 0. \end{cases}$$

Evidently τ is Lipschitz continuous. Let us set $j_1(z,\zeta) \stackrel{df}{=} j(z,\tau(\zeta))$. For almost all $z\in Z$, $j_1(z,\cdot)$ is locally Lipschitz and from Clarke [7, p. 42], we know that

$$\partial j_1(z,\zeta) \subseteq \begin{cases} 0 & \text{if } \ \zeta < 0 \\ \operatorname{conv}\{\lambda\partial j(z,0)\colon\ \lambda\in[0,1]\} & \text{if } \ \zeta = 0 \\ \partial j(z,\zeta) & \text{if } \ \zeta > 0. \end{cases} \tag{3.1}$$

Let $\varphi_1\colon W_0^{1,p}(Z) \longrightarrow \mathbb{R}$ be the energy functional, defined by

$$\varphi_1(x) \stackrel{df}{=} \frac{1}{p}\|\nabla x\|_p^p - \int_Z j_1\big(z,x(z)\big)\mathrm{d}z \quad \forall x\in W_0^{1,p}(Z).$$

We know that φ_1 is locally Lipschitz (see Hu and Papageorgiou [18, p. 313]). We start our study of problem (1.1) with a simple observation.

LEMMA 3.2. *There exist* $\xi > 0$, *such that*

$$\|\nabla x\|_p^p - \int_Z \vartheta_0(z)|x(z)|^p\mathrm{d}z \geqslant \xi\|\nabla x\|_p^p \quad \forall x\in W_0^{1,p}(Z).$$

Proof. Let $\psi\colon W_0^{1,p}(Z) \longrightarrow \mathbb{R}$ be defined by

$$\psi(x) \stackrel{df}{=} \|\nabla x\|_p^p - \int_Z \vartheta_0(z)|x(z)|^p\mathrm{d}z \quad \forall x\in W_0^{1,p}(Z).$$

Since by hypothesis $\vartheta_0(z) \leqslant \lambda_1$ for almost all $z\in Z$, from the variational characterization of $\lambda_1 > 0$ (see (2.2)), we have $\psi \geqslant 0$.

Suppose that the lemma is not true. Then by the homogeneity of ψ, we can find a sequence $\{x_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(Z)$ with $\|\nabla x_n\|_p = 1$ for $n\geqslant 1$, such that $\psi(x_n) \searrow 0$.

By the Poincaré inequality and by passing to a subsequence if necessary, we may assume that

$$x_n \xrightarrow{w} x \quad \text{in } W_0^{1,p}(Z)$$
$$x_n \longrightarrow x \quad \text{in } L^p(Z)$$
$$x_n(z) \longrightarrow x(z) \quad \text{for a.a. } z \in Z$$
$$|x_n(z)| \leqslant k(z) \quad \text{for a.a. } z \in Z,$$

with $k \in L^p(Z)$. Then, from the weak lower semicontinuity of the norm functional, we have

$$\|\nabla x\|_p^p \leqslant \liminf_{n \to +\infty} \|\nabla x_n\|_p^p$$

and

$$\int_Z \vartheta_0(z)|x_n(z)|^p \mathrm{d}z \longrightarrow \int_Z \vartheta_0(z)|x(z)|^p \mathrm{d}z.$$

So in the limit as $n \to +\infty$, we obtain

$$\|\nabla x\|_p^p - \int_Z \vartheta_0(z)|x(z)|^p \mathrm{d}z \leqslant 0. \tag{3.2}$$

But $\vartheta_0(z) \leqslant \lambda_1$ for almost all $z \in Z$, so

$$\|\nabla x\|_p^p \leqslant \lambda_1 \|x\|_p^p$$

and so from (2.2), we get that $x = \pm u_1$ or $x \equiv 0$.

If $x \equiv 0$, the $\|\nabla x_n\|_p \longrightarrow 0$, hence from the Poincaré inequality

$$x_n \longrightarrow 0 \quad \text{in } W_0^{1,p}(Z),$$

a contradiction to the fact that $\|\nabla x_n\|_p = 1$ for $n \geqslant 1$. So $x = \pm u$. Since $u_1(z) > 0$ for all $z \in Z$ (see Section 2), from (3.2), we have

$$\|\nabla x\|_p^p \leqslant \int_Z \vartheta_0(z)|x(z)|^p \mathrm{d}z < \lambda_1 \|x\|_p^p \tag{3.3}$$

(recall that the inequality $\vartheta_0(z) \leqslant \lambda_1$ is strict on a set of positive measure). Comparing (2.2) and (3.3) we reach a contradiction. This proves the lemma. □

PROPOSITION 3.3. *If hypotheses $H(j)$ hold, then φ_1 satisfies the nonsmooth Cerami condition at any level $c \geqslant 0$.*

Proof. Let $c \geqslant 0$ and consider a sequence $\{x_n\}_{n \geqslant 1} \subseteq W_0^{1,p}(Z)$, such that

$$\varphi_1(x_n) \longrightarrow c \quad \text{and} \quad \big(1 + \|x_n\|_{W^{1,p}(Z)}\big) m_{\varphi_1}(x_n) \longrightarrow 0 \quad \text{as } n \to +\infty.$$

Let $x_n^* \in \partial\varphi_1(x_n)$ be such that $m_{\varphi_1}(x_n) = \|x_n^*\|_{W^{-1,p'}(Z)}$. Such an element exists since the set $\partial\varphi_1(x_n) \subseteq W^{-1,p'}(Z) = (W_0^{1,p}(Z))^*$ is weakly compact (see Section 2) and the norm functional is weakly lower semicontinuous. Let $A\colon W_0^{1,p}(Z) \longrightarrow W^{-1,p'}(Z)$ be the nonlinear operator, defined by

$$\langle A(x), y\rangle_{W_0^{1,p}(Z)} = \int_Z \|\nabla x(z)\|_{\mathbb{R}^N}^{p-2} (\nabla x(z), \nabla y(z))_{\mathbb{R}^N} \mathrm{d}z \quad \forall x, y \in W_0^{1,p}(Z).$$

It is easy to check that A is monotone and demicontinuous, thus it is maximal monotone (see Denkowski et al. [10, Corollary 1.3.15, p. 37] or Hu and Papageorgiou [17, Corollary III.1.35, p. 309]). We have

$$x_n^* = A(x_n) - u_n \quad \forall n \geqslant 1, \tag{3.4}$$

with $u_n \in L^{p'}(Z)$ (where $\frac{1}{p} + \frac{1}{p'} = 1$), $u_n(z) \in \partial j_1(z, x_n(z))\mathrm{d}z$ (see Clarke [7, p. 80]). From the choice of the sequence $\{x_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(Z)$, we have

$$\left|\langle x_n^*, x_n\rangle_{W_0^{1,p}(Z)}\right| \leqslant \varepsilon_n \quad \forall n \geqslant 1, \tag{3.5}$$

with $\varepsilon_n \searrow 0$ and so from (3.4), we get

$$\left| \|\nabla x_n\|_p^p - \int_Z x_n(z) u_n(z) \mathrm{d}z \right| \leqslant \varepsilon_n \quad \forall n \geqslant 1. \tag{3.6}$$

We claim that the sequence $\{x_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(Z)$ is bounded. Suppose that this is not the case. Then, by passing to a subsequence if necessary, we may assume that $\|x_n\|_{W^{1,p}(Z)} \longrightarrow +\infty$. If we use as test function $x_n^- \in W_0^{1,p}(Z)$ (see Section 2) and we take into account (3.1), we obtain

$$\left| \|\nabla x_n^-\|_p^p - \int_Z x_n^-(z) u_n(z) \mathrm{d}z \right| = \|\nabla x_n^-\|_p^p \leqslant \varepsilon_n$$

and by the Poincaré inequality, we have

$$x_n^- \longrightarrow 0 \quad \text{in } W_0^{1,p}(Z).$$

Since

$$\|x_n\|_{W^{1,p}(Z)} = \|x_n^+\|_{W^{1,p}(Z)} + \|x_n^-\|_{W^{1,p}(Z)} \quad \forall n \geqslant 1,$$

it follows that $\|x_n^+\|_{W^{1,p}(Z)} \longrightarrow +\infty$. This means that in order to prove the boundedness of $\{x_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(Z)$ we need to concentrate on $\{x_n^+\}_{n\geqslant 1} \subseteq W_0^{1,p}(Z)$ since $x_n^- \to 0$ in $W_0^{1,p}(Z)$. Indeed note that from (3.6) we have $\left| \|\nabla x_n^+\|_p^p - \int_Z x_n^+(z) u_n(z) \mathrm{d}z \right| \leqslant \varepsilon_n + \|\nabla x_n^-\|_p^p = \widehat{\varepsilon}_n$ with $\widehat{\varepsilon}_n \searrow 0$. Thus it is the part x_n^+ that matters. So without any loss of generality, we may assume that $x_n \geqslant 0$ for all $n \geqslant 1$. Let us set

$$y_n \stackrel{df}{=} \frac{x_n}{\|x_n\|_{W^{1,p}(Z)}} \quad \forall n \geqslant 1. \tag{3.7}$$

By passing to a subsequence if necessary, we may assume that

$$\begin{aligned}
&y_n \xrightarrow{w} y \quad \text{in } W_0^{1,p}(Z)\\
&y_n \longrightarrow y \quad \text{in } L^p(Z)\\
&y_n(z) \longrightarrow y(z) \quad \text{for a.a. } z\in Z\\
&|y_n(z)| \leqslant k(z) \quad \text{for a.a. } z\in Z,
\end{aligned}$$

with $k\in L^p(Z)$. We show that $y\not\equiv 0$. Suppose that $y\equiv 0$. By virtue of hypothesis $H(j)$(v), for a given $\varepsilon>0$, we can find $M_1=M_1(\varepsilon)>0$, such that

$$0 \leqslant \frac{u}{\zeta^{p-1}} \leqslant \vartheta_1(z)+\varepsilon \quad \text{for a.a. } z\in Z, \text{ all } \zeta\geqslant M_1, \; u\in\partial j(z,\zeta)=\partial j_1(z,\zeta).$$

In addition, from hypotheses $H(j)$ (ii), (iii) and (iv), we have

$$0 \leqslant \frac{u}{\zeta^{p-1}} \leqslant \hat{k}(z) \quad \text{for a.a. } z\in Z \text{ all } \zeta\in(0,M_1], \; u\in\partial j(z,\zeta),$$

where $\hat{k}(z)\overset{df}{=}\dfrac{k_{M_1}(z)}{M^{p-1}}$. So finally, we can say that

$$0 \leqslant \frac{u}{\zeta^{p-1}} \leqslant k_0(z) \quad \text{for a.a. } z\in Z \text{ all } \zeta>0, \; u\in\partial j(z,\zeta)=\partial j_1(z,\zeta), \tag{3.8}$$

with $k_0\in L^\infty(Z)$. Dividing (3.6) by $\|x_n\|^p_{W^{1,p}(Z)}$, we obtain

$$\left| \|\nabla y_n\|_p^p - \int_Z \frac{u_n(z)}{\|x_n\|^{p-1}_{W^{1,p}(Z)}} y_n(z)\mathrm{d}z \right| \leqslant \frac{\varepsilon_n}{\|x_n\|^p_{W^{1,p}(Z)}},$$

so

$$\left| \|\nabla y_n\|_p^p - \int_{Z_n} \frac{u_n(z)}{\|x_n\|^{p-1}_{W^{1,p}(Z)}} y_n(z)\mathrm{d}z \right| \leqslant \frac{\varepsilon_n}{\|x_n\|^p_{W^{1,p}(Z)}},$$

where $Z_n\overset{df}{=}\{z\in Z: x_n(z)>0\}$ (recall that $x_n\geqslant 0$ for $n\geqslant 1$). Thus from (3.7), we get

$$\left| \|\nabla y_n\|_p^p - \int_{Z_n} \frac{u_n(z)}{\big(x_n(z)\big)^{p-1}} \big(y_n(z)\big)^p\mathrm{d}z \right| \leqslant \frac{\varepsilon_n}{\|x_n\|^p_{W^{1,p}(Z)}}$$

and from (3.8), we have

$$\begin{aligned}
\|\nabla y_n\|_p^p &\leqslant \frac{\varepsilon_n}{\|x_n\|^p_{W^{1,p}(Z)}} + \int_Z k_0(z)\big(y_n(z)\big)^p\mathrm{d}z\\
&\leqslant \frac{\varepsilon_n}{\|x_n\|^p_{W^{1,p}(Z)}} + \|k_0\|_\infty\|y_n\|_p^p,
\end{aligned}$$

so

$$y_n \longrightarrow 0 \quad \text{in } W_0^{1,p}(Z)$$

(recall that we have assumed that $y \equiv 0$). But this contradicts the fact that $\|y_n\|_{W^{1,p}(Z)} = 1$ for $n \geqslant 1$. So we infer that $y \not\equiv 0$.

Now, we introduce the following sequence of $L^\infty(Z)$-functions:

$$h_n(z) \stackrel{df}{=} \begin{cases} \dfrac{u_n(z)}{\big(x_n(z)\big)^{p-1}} & \text{if } z \in Z_n \\ 0 & \text{if } z \in Z \setminus Z_n \end{cases} \qquad \forall n \geqslant 1.$$

By virtue of hypotheses $H(j)$ (iii), (iv) and (v), we have

$$0 \leqslant h_n(z) \leqslant \vartheta_1(z) \quad \text{for a.a. } z \in Z \text{ and all } n \geqslant 1.$$

Thus, by passing to a subsequence if necessary, we may assume that

$$h_n \stackrel{w^*}{\longrightarrow} h \quad \text{in } L^\infty(Z)$$

with $h \in L^\infty(Z)$, such that

$$0 \leqslant h(z) \leqslant \vartheta_1(z) \quad \text{for a.a. } z \in Z. \tag{3.9}$$

Since we have assumed that $\|x_n\|_{W^{1,p}(Z)} \longrightarrow +\infty$ and because $y \not\equiv 0$, we must have that

$$x_n(z) \longrightarrow +\infty \quad \text{for a.a. } t \in \{y > 0\}.$$

From the choice of the sequence $\{x_n\}_{n \geqslant 1} \subseteq W_0^{1,p}(Z)$, we have

$$\left| \langle A(x_n), y_n - y \rangle_{W_0^{1,p}(Z)} - \int_Z u_n(z)\big(y_n - y\big)(z)\mathrm{d}z \right| \leqslant \varepsilon_n' \quad \forall n \geqslant 1,$$

with $\varepsilon_n' \searrow 0$ and so for $n \geqslant 1$, we have

$$\left| \langle A(y_n), y_n - y \rangle_{W_0^{1,p}(Z)} - \int_{Z_n} h_n(z)\big(y_n(z)\big)^{p-1}\big(y_n - y\big)(z)\mathrm{d}z - \right.$$
$$\left. - \int_{Z \setminus Z_n} \frac{u_n(z)}{\|x_n\|_{W_0^{1,p}(Z)}}(y_n - y)(z)\mathrm{d}z \right| \leqslant \frac{\varepsilon_n'}{\|x_n\|_{W^{1,p}(Z)}^{p-1}}.$$

Note that

$$\int_{Z_n} h_n(z)\big(y_n(z)\big)^{p-1}\big(y_n - y\big)(z)\mathrm{d}z \longrightarrow 0$$

and

$$\int_{Z\setminus Z_n} \frac{u_n(z)}{\|x_n\|_{W_0^{1,p}(Z)}}(y_n-y)(z)\mathrm{d}z \longrightarrow 0,$$

so

$$\limsup_{n\to+\infty}\langle A(y_n), y_n-y\rangle_{W_0^{1,p}(Z)} \leqslant 0. \tag{3.10}$$

But recall that A being maximal monotone and everywhere defined, it is generalized pseudomonotone (see Denkowski et al. [10, Corollary 1.3.67, p. 60] or Hu and Papageorgiou [17, Remark III.6.3, p. 365]). So from (3.10), it follows that

$$\langle A(y_n), y_n\rangle_{W_0^{1,p}(Z)} \longrightarrow \langle A(y), y\rangle_{W_0^{1,p}(Z)}$$

and so

$$\|\nabla y_n\|_p \longrightarrow \|\nabla y\|_p.$$

Because $\nabla y_n \xrightarrow{w} \nabla y$ in $L^p(Z;\mathbb{R}^N)$ and the latter space is uniformly convex (recall that $p\in(1,+\infty)$), we infer that

$$\nabla y_n \longrightarrow \nabla y \quad \text{in } L^p(Z;\mathbb{R}^N)$$

(Kadec–Klee property; see e.g. Denkowski et al. [9, Definition 3.6.32 and Proposition 3.6.33, p. 309]). Therefore

$$y_n \longrightarrow y \quad \text{in } W_0^{1,p}(Z).$$

Again from the choice of the sequence $\{x_n\}_{n\geqslant 1}\subseteq W_0^{1,p}(Z)$, for fixed $v\in W_0^{1,p}(Z)$, we have

$$\Big|\langle A(y_n), v\rangle_{W_0^{1,p}(Z)} - \int_{Z_n} h_n(z)\big(y_n(z)\big)^{p-1}v(z)\mathrm{d}z - \\ - \int_{Z\setminus Z_n}\frac{u_n(z)}{\|x_n\|_{W_0^{1,p}(Z)}}v(z)\mathrm{d}z\Big| \leqslant \frac{\varepsilon_n''(v)}{\|x_n\|^{p-1}_{W^{1,p}(Z)}},$$

with $\varepsilon_n''(v)\searrow 0$ and so

$$\Big|\int_Z \|\nabla y_n(z)\|^{p-2}_{\mathbb{R}^N}\big(\nabla y_n(z),\nabla v(z)\big)_{\mathbb{R}^N}\mathrm{d}z - \\ - \int_Z h_n(z)\big(y_n(z)\big)^{p-1}v(z)\mathrm{d}z\Big| \leqslant \frac{\varepsilon_n'''(v)}{\|x_n\|^{p-1}_{W^{1,p}(Z)}},$$

with $\varepsilon_n''' \searrow 0$ since $-\int_{Z\setminus Z_n} \frac{u_n(z)}{\|x_n\|_{W_0^{1,p}(Z)}} v(z)\mathrm{d}z \to 0$ as $n\to\infty$. Passing to the limit as $n\to+\infty$, we obtain

$$\int_Z \|\nabla y(z)\|_{\mathbb{R}^N}^{p-2}(\nabla y(z),\nabla v(z))_{\mathbb{R}^N}\mathrm{d}z = \int_Z h(z)(y(z))^{p-1}v(z)\mathrm{d}z \quad \forall v\in W_0^{1,p}(Z)$$

and so

$$\begin{cases} -\mathrm{div}(\|\nabla y(z)\|_{\mathbb{R}^N}^{p-2}\nabla y(z)) = h(z)(y(z))^{p-1} & \text{for a.a. } z\in Z \\ y|_\Gamma = 0. \end{cases} \tag{3.11}$$

From nonlinear regularity theory (see Lieberman [21]), we have that $y\in C^{1,\beta}(\overline{Z})$ with some $\beta\in(0,1)$ and so $y(z)\geqslant 0$ for all $z\in Z$, $y\not\equiv 0$. Invoking Theorem 5 of Vazquez [28], we obtain that $y(z)>0$ for all $z\in Z$. From (3.11), (3.9) and the fact that $\vartheta_1(z)\leqslant\lambda_1$ for almost all $t\in T$ with strict inequality on a set of positive measure, we have

$$\begin{aligned} \|\nabla y\|_p^p &= \int_Z h(z)|y(z)|^p\mathrm{d}z \leqslant \int_Z \vartheta_1(z)|y(z)|^p\mathrm{d}z \\ &< \lambda_1\|y\|_p^p \leqslant \|\nabla y\|_p^p, \end{aligned}$$

a contradiction, which proves the boundedness of the sequence $\{x_n\}_{n\geqslant 1}\subseteq W_0^{1,p}(Z)$. Hence, passing to a subsequence if necessary, we may assume that

$$\begin{aligned} x_n &\xrightarrow{w} x \quad \text{in } W_0^{1,p}(Z) \\ x_n &\longrightarrow x \quad \text{in } L^p(Z). \end{aligned}$$

Therefore, since

$$\left|\langle A(x_n), x_n - x\rangle_{W_0^{1,p}(Z)} - \int_Z u_n(z)(x_n-x)(z)\mathrm{d}z\right| \leqslant \varepsilon_n''' \quad \forall n\geqslant 1,$$

with $\varepsilon_n'''\searrow 0$, we obtain

$$\langle A(x_n), x_n - x\rangle_{W_0^{1,p}(Z)} \longrightarrow 0$$

and so as above, we conclude that

$$x_n \longrightarrow x \quad \text{in } W_0^{1,p}(Z).$$

□

Our aim is to verify that the energy functional φ_1 satisfies a ‘Mountain Pass geometry’ and so eventually apply Theorem 2.1. A first step in this direction is made by the next proposition.

PROPOSITION 3.4. *If hypotheses $H(j)$ hold, then there exist $\varrho,\beta>0$, such that*

$$\varphi_1(x) \geqslant \beta \quad \forall x\in W_0^{1,p}(Z),\ \|x\|_{W^{1,p}(Z)} = \varrho.$$

Proof. Let $\varepsilon \in (0, \xi\lambda_1)$, where $\xi > 0$ is as postulated in Lemma 3.2. By virtue of hypothesis $H(j)(v)$, we can find $\delta > 0$, such that

$$j_1(z,\zeta) = j(z,\zeta) \leqslant \frac{1}{p}(\vartheta_0(z)+\varepsilon)\zeta^p \quad \text{for a.a. } z \in Z \text{ and all } \zeta \in (0,\delta]. \tag{3.12}$$

On the other hand, from the other asymptotic condition (at $+\infty$) in hypothesis $H(j)(v)$, we can find $M_2 > 0$, such that

$$|u| \leqslant \big(\vartheta_1(z)+1\big)\zeta^{p-1} \quad \text{for a.a. } z \in Z \text{ and all } \zeta \geqslant M_2, u \in \partial j(z,\zeta) = \partial j_1(z,\zeta)$$

(see 3.1)). From the Lebourg mean value theorem (see Clarke [7, p. 41] or Denkowski et al. [9, Theorem 5.6.25, p. 609]), we know that we can find $\widehat{u}(z) \in \partial j(z,\zeta') = \partial j_1(z,\zeta')$, where $\zeta' = (1-\lambda)\zeta + \lambda M_2$ ($\zeta \geqslant M_2$ and $\lambda \in (0,1)$), such that

$$j_1(z,\zeta) - j_1(z,M_2) = \widehat{u}(z)(\zeta - M_2)$$

and so

$$j_1(z,\zeta) \leqslant j_1(z,M_2) + \big(\vartheta_1(z)+1\big)\zeta^p \tag{3.13}$$

By p^* we denote the critical Sobolev exponent, defined by

$$p^* \overset{df}{=} \begin{cases} \dfrac{Np}{N-p} & \text{if } p < N \\ +\infty & \text{if } p \geqslant N. \end{cases}$$

From (3.12) and (3.13) and hypothesis $H(j)$ (ii), it follows that we can find $\xi_1 > 0$ and $\eta \in (p, p^*]$ (both not depending on $\varepsilon > 0$), such that

$$j_1(z,\zeta) \leqslant \frac{1}{p}\big(\vartheta_0(z)+\varepsilon\big)|\zeta|^p + \xi_1|\zeta|^\eta \quad \text{for a.a. } z \in Z \text{ and all } \zeta \in \mathbb{R}. \tag{3.14}$$

Hence, we have

$$\begin{aligned} \varphi_1(x) &= \frac{1}{p}\|\nabla x\|_p^p - \int_Z j_1(z,x(z))\mathrm{d}z \\ &\geqslant \frac{1}{p}\|\nabla x\|_p^p - \frac{1}{p}\int_Z \vartheta_0(z)|x(z)|^p \mathrm{d}z - \\ &\quad - \frac{\varepsilon}{p}\|x\|_p^p - \xi_2\|\nabla x\|_p^\eta \quad \forall x \in W_0^{1,p}(Z), \end{aligned} \tag{3.15}$$

for some $\xi_2 > 0$ (not depending on ε). In obtaining the last inequality, we have used the Poincaré inequality and the Sobolev embedding theorem since $\eta \leqslant p^*$

(see Denkowski et al. [9, Theorem 3.9.52, p. 359]). From Lemma 3.2, we know that

$$\frac{1}{p}\|\nabla x\|_p^p - \frac{1}{p}\int_Z \vartheta_0(z)|x(z)|^p \mathrm{d}z \geqslant \frac{\xi}{p}\|\nabla x\|_p^p \quad \forall x \in W_0^{1,p}(Z). \tag{3.16}$$

Thus, using (3.16) and (2.2) in (3.15), we obtain

$$\varphi_1(x) \geqslant \frac{\xi}{p}\|\nabla x\|_p^p - \frac{\varepsilon}{\lambda_1 p}\|\nabla x\|_p^p - \xi_2\|\nabla x\|_p^\eta.$$

From the choice of $\varepsilon > 0$ (recall that $\varepsilon \in (0, \xi\lambda_1)$), we get

$$\varphi_1(x) \geqslant \xi_3\|\nabla x\|_p^p - \xi_2\|\nabla x\|_p^\eta \quad \forall x \in W_0^{1,p}(Z),$$

for some $\xi_3 > 0$. Since $\eta > p$ and using the Poincaré inequality, we can find $\varrho > 0$ small enough, so that

$$\varphi_1(x) \geqslant \beta > 0 \quad \forall x \in W_0^{1,p}(Z),\ \|x\|_{W^{1,p}(Z)} = \varrho.$$

This completes the proof of the proposition. □

PROPOSITION 3.5. *If hypotheses $H(j)$ hold, then $\varphi_1(tu_1) \longrightarrow -\infty$ as $t \to +\infty$.*

Proof. By hypothesis $H(j)(v)$, we know that $\lambda_1 \leqslant p\vartheta_1(z)$ for almost all $z \in Z$ with strict inequality on a set of positive measure. Since $u_1(z) > 0$ for all $z \in Z$, we have that

$$\int_Z \vartheta_1(z)\big(u_1(z)\big)^p \mathrm{d}z > \frac{\lambda_1}{p}\|u_1\|_p^p$$

and so

$$\int_Z \vartheta_1(z)\big(u_1(z)\big)^p \mathrm{d}z = \gamma + \frac{\lambda_1}{p}\|u_1\|_p^p, \tag{3.17}$$

for some $\gamma > 0$.

For $t > 0$, we have

$$\begin{aligned}\varphi_1(tu_1) &= \frac{t^p}{p}\|\nabla u_1\|_p^p - \int_Z j_1\big(z, tu_1(z)\big)\mathrm{d}z \\ &= \frac{t^p}{p}\|\nabla u_1\|_p^p - \int_Z j\big(z, tu_1(z)\big)\mathrm{d}z\end{aligned}$$

(since $u_1(z) > 0$ for all $z \in Z$).

Let $\varepsilon \in (0, \gamma)$. By virtue of hypothesis $H(j)(v)$, we can find $M_3 = M_3(\varepsilon) > 0$, such that

$$u \geqslant \big(\vartheta_1(z) - \varepsilon\big)\zeta^{p-1} \quad \text{for a.a. } z \in Z,\ \text{all } \zeta \geqslant M_3, \quad u \in \partial j(z,\zeta) = \partial j_1(z,\zeta)$$

(see (3.1)). Then as before using the Lebourg mean value theorem and the above inequality, we see that for almost all $z \in Z$ and all $\zeta \geqslant M_3$, we have

$$\begin{aligned} j(z,\zeta) - j(z,M_3) &= j_1(z,\zeta) - j_1(z,M_3) \\ &\geqslant \big(\vartheta_1(z) - \varepsilon\big)(\zeta - M_3)\zeta^{p-1} \geqslant \big(\vartheta_1(z) - \varepsilon\big)\zeta^p - \xi_4, \end{aligned}$$

for some $\xi_4 > 0$. Thus we can find $\xi_5 > 0$, such that

$$j(z,\zeta) \;=\; j_1(z,\zeta) \geqslant \big(\vartheta_1(z) - \varepsilon\big)\zeta^p - \xi_5 \quad \text{for a.a. } z \in Z \text{ and all } \zeta \geqslant 0.$$

So for all $t \in \mathbb{R}_+$, we have

$$\varphi_1(tu_1) \;\leqslant\; \frac{t^p}{p}\|\nabla u_1\|_p^p - t^p \int_Z \vartheta_1(z)\big(u_1(z)\big)^p \mathrm{d}z + \varepsilon t^p \|u_1\|_p^p + \xi_6, \tag{3.18}$$

for some $\xi_6 > 0$. Using (3.17), (3.18) and the fact that $\lambda_1 \|u_1\|_p^p = \|\nabla u_1\|_p^p$, we have

$$\varphi_1(tu_1) \;\leqslant\; \frac{t^p}{p}\|\nabla u_1\|_p^p - t^p\gamma - \frac{t^p}{p}\|\nabla u_1\|_p^p + \varepsilon t^p \|u_1\|_p^p + \xi_6.$$

As $\|u_1\|_p = 1$, we obtain

$$\varphi_1(tu_1) \;\leqslant\; -t^p\gamma + \varepsilon t^p + \xi_6 \;=\; t^p(-\gamma + \varepsilon) + \xi_6.$$

From the choice of $\varepsilon > 0$, we see that $\varphi_1(tu_1) \longrightarrow -\infty$ as $t \to +\infty$. □

Now the geometry is in place to apply the Nonsmooth Mountain Pass Theorem (see Theorem 2.1) and produce a critical point of φ_1, which we show that it is a positive solution of (1.1).

THEOREM 3.6. *If hypotheses $H(j)$ hold, then problem* (1.1) *has at least one positive solution which belongs in $C^1(\overline{Z})$.*

Proof. Propositions 3.3, 3.4 and 3.5 permit the application of Theorem 2.1 with $x_0 \overset{df}{=} 0$ (note that $\varphi_1(0) = 0$), $x_1 \overset{df}{=} t_1 u_1$, where $t_1 > 0$ is large enough, so that $\varphi(x_1) = \varphi_1(t_1 u_1) < 0$ (see Proposition 3.5) and $U \overset{df}{=} \big\{x \in W_0^{1,p}(Z) :\ \|x\|_{W^{1,p}(Z)} < \varrho\big\}$ (see Proposition 3.4). So we obtain $x_0 \in W_0^{1,p}(Z)$, such that

$$\varphi_1(x_0) \geqslant \beta > 0 \quad \text{and} \quad 0 \in \partial\varphi_1(x_0).$$

Hence, we have that

$$A(x_0) \;=\; u_0,$$

with $u_0 \in L^{p'}(Z)$, $u_0(z) \in \partial j_1\big(z, x_0(z)\big)$ for almost all $z \in Z$, $x_0 \not\equiv 0$. So, from (3.1), we have

$$\big\langle A(x_0), x_0^- \big\rangle_{W_0^{1,p}(Z)} \;=\; \int_Z u_0(z) x_0^-(z) \mathrm{d}z \;=\; 0$$

so

$$\|\nabla x_0^-\|_p^p = 0$$

and thus $x_0^- \equiv 0$. It follows that $x_0(z) \geqslant 0$ for almost all $z \in Z$ and $x_0 \not\equiv 0$. Also, we have

$$\langle A(x_0), \vartheta \rangle_{W_0^{1,p}(Z)} = \int_Z u_0(z)\vartheta(z)\mathrm{d}z \quad \forall \vartheta \in C_0^\infty(Z).$$

Using the Green identity and the fact that

$$-\mathrm{div}\big(\|\nabla x_0(\cdot)\|_{\mathbb{R}^N}^{p-2} \nabla x_0(\cdot)\big) \in W^{-1,p'}(Z) = \big(W_0^{1,p}(Z)\big)^*$$

(see Denkowski et al. [9, p. 362]), we obtain

$$\big\langle -\mathrm{div}\big(\|\nabla x_0(\cdot)\|_{\mathbb{R}^N}^{p-2} \nabla x_0(\cdot)\big), \vartheta \big\rangle_{W_0^{1,p}(Z)} = \int_Z u_0(z)\vartheta(z)\mathrm{d}z \quad \forall \vartheta \in C_0^\infty(Z).$$

Since the embedding $C_0^\infty(Z) \subseteq W_0^{1,p}(Z)$ is dense, it follows that

$$\begin{cases} -\mathrm{div}\big(\|\nabla x_0(z)\|_{\mathbb{R}^N}^{p-2} \nabla x_0(z)\big) = u_0(z) & \text{for a.a. } z \in Z \\ x_0|_\Gamma = 0. \end{cases}$$

From this and nonlinear regularity theory (see Ladyzhenskaya and Uraltseva [20, p. 286] and Lieberman [21]), we obtain that $x_0 \in C^{1,\beta}(\overline{Z})$ with some $\beta \in (0,1)$. Finally since $u_0(z) \geqslant 0$ for almost all $z \in Z$ (see hypothesis $H(j)$(iii) and recall that $x_0(z) \geqslant 0$ for all $z \in Z$), from Theorem 5 of Vazquez [28], we conclude that $x_0(z) > 0$ for all $z \in Z$ and $\frac{\partial x_0}{\partial n}(z) < 0$ for all $z \in \Gamma$ (here n denotes unit outward normal). □

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